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HOW MUCH ARE INCREASING SETS POSITIVELY CORRELATED?

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Two upward directed sets of sequences of zeroes and ones are positively correlated. We provide a lower bound on the correlation, in function of how much the two sets simultaneously depend on the same coordinates.

I. Introduction

Consider a set A of sequences of length n of zeroes and ones, that is, $A \subset \{0,1\}^n$. We say that A is increasing (or upward directed) if

$$x \in A, \forall i \leq n, y_i \geq x_i \Rightarrow y \in A.$$

Given two increasing subsets A, B of $\{0,1\}^n$, a well known result asserts that

$$2^n \text{card } (A \cap B) \geq \text{card } A \text{card } B$$

or, equivalently

$$\mu(A \cap B) \geq \mu(A)\,\mu(B) \tag{1.1}$$

where $\mu(A) = 2^{-n} \text{card } A$ is the normalized counting measure on $\{0,1\}^n$.

For $x \in \{0,1\}^n$, and $i \leq n$, consider the sequence $T_i x$ obtained from x by changing the i^{th}-coordinate. For a set $A \subset \{0,1\}^n$, we set

$$A_i = \{x \in A; T_i x \notin A\}.$$

The quantity $\mu(A_i)$ expresses "how much A depends on the i^{th} coordinate." In particular if $\mu(A_i) = 0$, then A does not depend on the i^{th} coordinate in the sense

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that $x \in A$ if and only if $T_i x \in A$. We should also observe that if A is increasing then $x \in A_i \Rightarrow x_i = 1$.

The usual proof of (1.1) by induction upon the number of coordinates shows that equality holds in (1.1) if and only if for each coordinate i, either A or B does not depend on i. A natural question, which is the object of the present paper, is to find a quantitative version of this fact, that is to find a lower bound of $\mu(A \cap B) - \mu(A)\mu(B)$ in function of "how much A and B simultaneously depend on the same coordinates." Certainly there are many conceivable ways to quantify this, but, since A and B simultaneously depend on coordinate i if and only if $\mu(A_i)\mu(B_i) > 0$, it is natural to introduce these quantities. We will prove the following.

Theorem 1.1. *Consider, for $0 \le x \le 1$, the function $\varphi(x) = x/\log(e/x)$. Then, for some universal constant K, for all n and all increasing sets $A, B \subset \{0,1\}^n$, we have*

$$\mu(A \cap B) - \mu(A)\mu(B) \ge \frac{1}{K}\varphi\left(\sum_{i \le n} \mu(A_i)\mu(B_i)\right). \tag{1.2}$$

It would be of course very difficult to have an exact expression for the difference $\mu(A \cap B) - \mu(A)\mu(B)$. But we will show that in a case of special importance, the lower bound given by (1.2) is of correct order. Consider an integer $k \ge n/2$, and set

$$A = \left\{(x_i)\,;\ \sum_{i \le n} x_i \ge n - k\right\};\ B = \left\{(x_i)\,;\ \sum_{i \le n} x_i > k\right\}.$$

Thus $\mu(A) + \mu(B) = 1$. Set $\varepsilon = \mu(B)$. Since $B \subset A$, we have

$$\mu(A \cap B) - \mu(A)\mu(B) = \varepsilon - \varepsilon(1 - \varepsilon) = \varepsilon^2.$$

On the other hand computation (or the arguments of Section 2) show that $\sum_{i \le n} \mu(A_i)\mu(B_i)$ is of order $\varepsilon^2 \log(1/\varepsilon)$. And $\varphi\left(\varepsilon^2 \log(1/\varepsilon)\right)$ is of order ε^2, so that (1.2) is sharp in this case. This should be the moment to point out that the weaker inequality

$$\mu(A \cap B) - \mu(A)\mu(B) \ge \sum_{i \le n} \varphi(\mu(A_i)\mu(B_i)) \tag{1.3}$$

that is considerably easier to prove than (1.2), would not be sharp on the example above.

The proof of (1.2) will be by induction over the number of coordinates. However, before that can be done a new fact has to be proved about increasing sets. While somewhat technical, this new fact is the heart of the paper, and is better explained as part of a circle of ideas.

2. Harmonic analysis

The title of this section is choosen to pay respect to the remarkable work [1]. While our methods belong to the same circle of ideas, they actually are not connected very much with harmonic analysis, but rather with probability.

For $i \leq n$, let us consider the function $r_i(x) = 2x_i - 1$ on $\{0,1\}^n$. Thus $|r_i|=1, \int r_i d\mu=0$. If A is an increasing set, we observe that

$$\mu(A_i) = \int_A r_i d\mu. \tag{2.1}$$

Since the functions r_i constitute an orthonormal system in $L^2(\mu)$, we obtain immediately that for <u>any</u> set A, we have

$$\sum_{i\leq n} \left(\int_A r_i d\mu \right)^2 \leq \mu(A). \tag{2.2}$$

(In the case of increasing sets, this reads $\sum_{i\leq n} \mu(A_i)^2 \leq \mu(A)$.)

This is however not sharp when $\mu(A)$ is small, and a sharp inequality will be an essential tool.

Proposition 2.1 (Subgaussian inequality). (see e.g. [2] p. 90) *Consider numbers* $(\alpha_i)_{i\leq n}$. *Then*

$$\mu\left(\left| \sum_{i\leq n} \alpha_i r_i \right| \geq t \right) \leq 2\exp\left(-\frac{t^2}{2\sum_{i\leq n} \alpha_i^2} \right). \tag{2.3}$$

To simplify notations, we now denote by K a universal constant, not necessarily the same at each occurence.

Proposition 2.2. *For some universal constant K and any subset of $\{0,1\}^n$ we have*

$$\sum_{i\leq n} \left(\int_A r_i d\mu \right)^2 \leq K\mu(A)^2 \log \frac{e}{\mu(A)}. \tag{2.4}$$

Proof. Step 1. Consider numbers $(\alpha_i)_{i\leq n}$ with $\sum_{i\leq n} \alpha_i^2 = 1$, and set $f = \sum_{i\leq n} \alpha_i r_i$, so that, by (2.3), we have $\mu(|f|\geq t) \leq 2\exp(-t^2/2)$. Now, for any $t_0 \geq 1$

$$\int_A |f| d\mu = \int_0^\infty \mu(\{|f| \geq t\} \cap A)\, dt \leq \int_0^\infty \min\left(\mu(A), 2e^{-t^2/2}\right) dt$$

$$\leq \mu(A)\, t_0 + 2\int_{t_0}^{\infty} e^{-t^2/2}dt \leq \mu(A)\, t_0 + 2\int_{t_0}^{\infty} te^{-t^2/2}dt \leq \mu(A)\, t_0 + 2e^{-t_0^2/2}.$$

Taking $t_0 = \sqrt{2\log\frac{e}{\mu(A)}} \geq 1$ yields

$$\int_A f d\mu \leq K\mu(A)\sqrt{\log\frac{e}{\mu(A)}} \tag{2.5}$$

Step 2. Taking $\beta_i = \int_A r_i d\mu, \alpha_i = \beta_i \left(\sum_{i\leq n} \beta_i^2\right)^{-1/2}$ in (2.5) yield the result. ∎

For an increasing set A, a quantity that will be of importance in the proof of Theorem 1.1 is

$$H(A) = \sum_{k\leq n}\sum_{i\neq k}\left(\int_{A_k} r_i d\mu\right)^2.$$

To find a bound for $H(A)$ one can observe that A_k identifies to a subset of $\{0,1\}^{n-1}$ (since $x_k = 1$ for x in A_k) so that by (2.4)

$$\sum_{i\neq k}\left(\int_{A_k} r_i d\mu\right)^2 \leq K\mu(A_k)^2 \log\frac{e}{\mu(A_k)}.$$

By summation over k this yields

$$H(A) = \sum_{k\leq n} \mu(A_k)^2 \log\frac{e}{\mu(A_k)}. \tag{2.6}$$

The essential ingredient in proving (1.2) rather than (1.3) is the following improvement of (2.6).

Proposition 2.3. *For any increasing set*

$$H(A) \leq K\sum_{k\leq n}\mu(A_k)^2 \log\frac{K}{\sum_{k\leq n}\mu(A_k)^2}. \tag{2.7}$$

Let us comment on this result. The function $x\log\frac{K}{x}$ increases near zero. For increasing sets, (2.4) reads

$$\sum_{k\leq n}\mu(A_k)^2 \leq K\mu(A)^2\log\frac{K}{\mu(A)}$$

so that (2.7) implies

$$H(A) \le K\mu(A)^2 \left(\log \frac{K}{\mu(A)}\right)^2 . \tag{2.8}$$

Now it is easy to see that, for an increasing set A we have

$$\int_A r_i r_k d\mu = \int_{A_k} r_i d\mu$$

so that (2.8) reads

$$\sum_{i \neq k} \left(\int_A r_i r_k d\mu \right)^2 \le K\mu(A)^2 \left(\log \frac{K}{\mu(A)}\right)^2 . \tag{2.9}$$

This inequality, that should be compared to (2.4), is known to hold for any set A. It can be e.g. proved along the lines of (2.4), using now an exponential inequality such as (see e.g. [2] p. 105).

$$\mu\left(\left|\sum_{i\neq k} \alpha_{ik} r_i r_k\right| \ge t\right) \le K \exp\left(-\frac{t}{K\left(\sum_{i,k\le n} \alpha_{ik}^2\right)^{1/2}}\right).$$

Thus, (2.7) can also be described, as an improvement in the case of increasing sets, over the (semi) classical inequality (2.9).

For the proof of Theorem 1.1, we actually need the following "decoupled" version of Proposition 2.3, of which this Proposition is an immediate corollary.

Theorem 2.4. *Given two increasing sets A, B we have*

$$\sum_{k\le n} \sum_{i\neq k} \left|\int_{A_k} r_i d\mu\right| \left|\int_{B_k} r_i d\mu\right| \le K \sum_{k\le n} \mu(A_k)\,\mu(B_k) \log \frac{e}{\sum_{k\le n} \mu(A_k)\,\mu(B_k)}.$$

3. Proof of Theorem 2.4.

The main step of the proof of Theorem 2.4 is the following lemma.

Lemma 3.1. *Consider a partition I, J of $\{1,\dots,n\}$. Consider an increasing set A and $S>0$. Consider*

$$L = \{k \in J;\ \left(\sum_{i\in I}\left(\int_{A_k} r_i d\mu\right)^2\right)^{1/2} \geq S\mu(A_k)\}.$$

Then

$$\sum_{k\in L}\mu(A_k)^2 \leq K\exp\left(-\frac{S^2}{K}\right). \tag{3.1}$$

Proof. Step 1. Given $k\in L$, we find numbers $(\alpha_{i,k})_{i\in I}$ such that $\sum\limits_{i\in I}\alpha_{i,k}^2=1$ and

$$\int_{A_k}\sum_{i\in I}\alpha_{i,k}r_i d\mu = \left(\sum_{i\in I}\left(\int_{A_k} r_i d\mu\right)^2\right)^{1/2} \geq S\mu(A_k).$$

We set $f_k=\sum_{i\in I}\alpha_{i,k}r_i$, so that

$$\int_{A_k} f_k d\mu \geq S\mu(A_k). \tag{3.2}$$

Step 2. The main idea of using two disjoint sets I,J is that we can think to the basic space $\{0,1\}^n$ as a product $\Omega_1\times\Omega_2$, where $\Omega_1=\{0,1\}^I, \Omega_2=\{0,1\}^J$. Accordingly, we can write $\omega\in\{0,1\}^n$ as $(\omega_1,\omega_2), \omega_1\in\Omega_1, \omega_2\in\Omega_2$.

Given $\omega_1\in\Omega_1$, we consider the set $A_{\omega_1}\subset\Omega_2$ given by

$$A_{\omega_1} = \{\omega_2\in\Omega_2; (\omega_1,\omega_2)\in A\}.$$

Clearly, this is an increasing set. Moreover, for $k\in J$, we have (with obvious notations) $(A_{\omega_1})_k = A_{k,\omega_1}$, where

$$A_{k,\omega_1} = \{\omega_2\in\Omega_2; (\omega_1,\omega_2)\in A_k\}.$$

We denote by μ_j the normalized counting measure on $\Omega_j\,(j=1,2)$. It follows from (2.2) that, for each $\omega_1\in\Omega_1$, we have

$$\sum_{k\in J}\mu_2^2(A_{k,\omega_1}) = \sum_{k\in J}\mu_2^2((A_{\omega_1})_k) \leq 1. \tag{3.3}$$

Given $\omega_1 \in \Omega_1, k$ in L, we observe that the function $f_k(\omega_1, \omega_2)$ does not depend on ω_2. We denote by $f_k(\omega_1)$ its value. Thus

$$\int 1_{A_k}(\omega_1, \omega_2) f_k(\omega_1, \omega_2) \, d\mu_2(\omega_2) = f_k(\omega_1) \mu_2(A_{k,\omega_1}). \tag{3.4}$$

We now integrate (3.4) in ω_1. Using Fubini theorem, we obtain

$$\int_{A_k} f_k \, d\mu = \int_{\Omega_1} f_k(\omega_1) \mu_2(A_{k,\omega_1}) \, d\mu_1(\omega_1). \tag{3.5}$$

Step 3. For $k \in L$, we define

$$\Omega_{k,0} = \{\omega_1; |f_k(\omega_1)| \leq 2\}$$

and for $p \geq 1$ we define

$$\Omega_{k,p} = \left\{\omega_1; 2^p < |f_k(\omega_1)| \leq 2^{p+1}\right\}.$$

We set

$$C_{k,p} = \int_{\Omega_{k,p}} f_k(\omega_1) \mu_2(A_{k,\omega_1}) \, d\mu_1(\omega_1).$$

Thus, by Cauchy–Schwarz, we have

$$\begin{aligned} C_{k,p}^2 &\leq \mu_1(\Omega_{k,p}) \int_{\Omega_{k,p}} f_k^2(\omega_1) \mu_2^2(A_{k,\omega_1}) \, d\mu_1(\omega_1) \\ &\leq \mu_1(\Omega_{k,p}) \, 2^{2p+2} \int_{\Omega_{k,p}} \mu_2^2(A_{k,\omega_1}) \, d\mu_1(\omega_1). \end{aligned}$$

Since $f_k = \sum_{i \in I} \alpha_{i,k} r_i$ where $\sum_{i \in I} \alpha_{i,k}^2 = 1$ we can appeal to (2.3) to get

$$\mu_1(\Omega_{k,p}) \leq 2 \exp\left(-2^{2p-1}\right)$$

so that

$$C_{k,p}^2 \leq 2^{2p+3} \exp\left(-2^{2p-1}\right) \int_{\Omega_1} \mu_2^2(A_{k,\omega_1}) \, d\mu_1(\omega_1).$$

Combining with (3.3), we get

$$\sum_{k \in L} C_{k,p}^2 \leq 2^{2p+3} \exp\left(-2^{2p-1}\right). \tag{3.6}$$

Step 4. It follows from (3.2) that

$$S\mu(A_k) \le \sum_{p\ge 0} C_{k,p}.$$

Consider an integer $p_0 \ge 0$ that will be determined later, and set $C_k = \sum_{p\le p_0} C_{k,p}$. Thus

$$S\mu(A_k) \le \sum_{p\ge 0} C_{k,p} \le C_k + \sum_{p>p_0} C_{k,p}. \tag{3.7}$$

We observe the elementary inequality

$$\left(\sum_{p\ge 1} \beta_p\right)^2 \le \sum_{p\ge 1} 2^p \beta_p^2$$

that follows from the convexity of the function $x \to x^2$ and the fact that $\sum_{p\ge 1} \beta_p = \sum_{p\ge 1} 2^{-p}\left(2^p\beta_p\right)$. Thus, from (3.7) we have

$$S^2\mu(A_k)^2 \le 2C_k^2 + \sum_{p>p_0} 2^p C_{k,p}^2. \tag{3.8}$$

We now sum over k in L, using (3.6), to get

$$S^2 \sum_{k\in L} \mu(A_k)^2 \le 2\sum_{k\in L} C_k^2 + \sum_{p>p_0} 2^{3p+3} \exp\left(-2^{2p-1}\right). \tag{3.9}$$

We now observe that $C_k \le 2^{p_0+2}\mu(A_k)$, so that

$$S^2 \sum_{k\in L} \mu(A_k)^2 \le 2^{2p_0+3} \sum_{k\in L} \mu(A_k)^2 + \sum_{p>p_0} 2^{3p+3} \exp\left(-2^{2p-1}\right). \tag{3.10}$$

We now choose p_0 to be the largest such that $2^{2p_0+3} \le S^2/2$. Thus $p_0 \ge 0$ whenever $S \ge 4$.

From (3.10) we get

$$S^2 \sum_{k\in L} \mu(A_k)^2 \le \sum_{p>p_0} 2^{3p+4} \exp\left(-2^{2p-1}\right). \tag{3.11}$$

We now observe that, by definition of p_0 we have $S^2 < 2^{2(p_0+1)+4}$. Also,

$$\sum_{p>p_0} 2^{3p+4} \exp\left(-2^{2p-1}\right) \le K \exp\left(-2^{2p_0}\right)$$

so that the result follows from (3.11) when $S \geq 4$ (and from (2.2) when $S < 4$). ∎

We now prove Theorem 2.4.

Step 1. The first observation is that it suffices to prove that given any partition I, J of $\{1, \ldots, n\}$, we have

$$\sum_{i \in I} \sum_{k \in J} \left| \int_{A_k} r_i d\mu \right| \left| \int_{B_k} r_i d\mu \right| \leq KU \log \frac{e}{U} \tag{3.12}$$

where

$$U = \sum_{k \leq n} \mu(A_k) \mu(B_k).$$

Indeed, Theorem 2.4 then follows by averaging over all possible choices of I, J.

Step 2. We show that if $S > 0$ and

$$L_S = \left\{ k \in J; \sum_{i \in I} \left| \int_{A_k} r_i d\mu \right| \left| \int_{B_k} r_i d\mu \right| \geq S^2 \mu(A_k) \mu(B_k) \right\}$$

then

$$\sum_{k \in L_S} \mu(A_k) \mu(B_k) \leq K \exp\left(-\frac{S^2}{K} \right). \tag{3.13}$$

Indeed, let us define

$$L^1 = \left\{ k \in J; \left(\sum_{i \in I} \left(\int_{A_k} r_i d\mu \right)^2 \right)^{1/2} \geq S \mu(A_k) \right\},$$

$$L^2 = \left\{ k \in J; \left(\sum_{i \in I} \left(\int_{A_k} r_i d\mu \right)^2 \right)^{1/2} \geq S \mu(B_k) \right\}.$$

It follows from Cauchy–Schwarz that $L \subset L^1 \cup L^2$. Using Cauchy–Schwarz again,

$$\sum_{k \in L^1} \mu(A_k) \mu(B_k) \leq \left(\sum_{k \in L^1} \mu(A_k)^2 \right)^{1/2} \left(\sum_{k \in L^1} \mu(B_k)^2 \right)^{1/2}.$$

We now appeal to Lemma 3.1 to bound the first sum, and to (2.2) to bound the second sum, and we get

$$\sum_{k \in L^1} \mu(A_k) \mu(B_k) \leq K \exp\left(-\frac{S^2}{K} \right).$$

A similar inequality for L^2 concludes the proof.

Step 3. We set

$$L_0 = \left\{ k \in J; \sum_{i \in I} \left| \int_{A_k} r_i d\mu \right| \left| \int_{B_k} r_i d\mu \right| \le 4\mu(A_k)\mu(B_k) \right\}.$$

For $p \ge 1$, we set

$$L_p = \left\{ k \in J; 2^{2p}\mu(A_k)\mu(B_k) < \sum_{i \in I} \left| \int_{A_k} r_i d\mu \right| \left| \int_{B_k} r_i d\mu \right| \le 2^{2p+2}\mu(A_k)\mu(B_k) \right\}.$$

Thus by (3.13) we have

$$\sum_{k \in L_p} \mu(A_k)\mu(B_k) \le K \exp\left(-\frac{2^{2p}}{K}\right),$$

and thus

$$\sum_{k \in L_p} \sum_{i \in I} \left| \int_{A_k} r_i d\mu \right| \left| \int_{B_k} r_i d\mu \right| \le K 2^{2p} \exp\left(-\frac{2^{2p}}{K}\right).$$

Consider now an integer p_0, to be determined later. Then

$$\sum_{p > p_0} \sum_{k \in L_p} \sum_{i \in I} \left| \int_{A_k} r_i d\mu \right| \left| \int_{B_k} r_i d\mu \right| \le K \sum_{p > p_0} 2^{2p} \exp\left(-\frac{2^{2p}}{K}\right)$$

$$\le K \exp\left(-\frac{2^{2p_0}}{K}\right). \tag{3.14}$$

Also, by definition of L_p,

$$\sum_{p \le p_0} \sum_{k \in L_p} \sum_{i \in I} \left| \int_{A_k} r_i d\mu \right| \left| \int_{B_k} r_i d\mu \right| \le 2^{2p_0+2} \sum_{k \in J} \mu(A_k)\mu(B_k)$$

$$\le 2^{2p_0+2} U.$$

Thus

$$\sum_{k \in J} \sum_{i \in I} \left| \int_{A_k} r_i d\mu \right| \left| \int_{B_k} r_i d\mu \right| \le 2^{2p_0+2} U + K \exp\left(-\frac{2^{2p_0}}{K}\right).$$

This result then follows by taking for p_0 the smallest integer ≥ 0 for which $\exp\left(-2^{2p_0}/K\right) \leq U$. ∎

4. Proof of Theorem 1.1

Let us start by some easy observations about the function φ.

Lemma 4.1. *If* $0 \leq u \leq v \leq 1$, *we have*

$$\varphi(v) \leq \varphi(u) + \frac{2(v-u)}{\log \frac{e}{v}}. \tag{4.1}$$

Proof. We have

$$\varphi'(x) = \frac{1}{\log \frac{e}{x}} + \frac{1}{\left(\log \frac{e}{x}\right)^2}$$

so that φ' increases, (and thus φ is convex) and $\varphi'(x) \leq 2/\log(e/x)$. By convexity, we then have

$$\varphi(v) \leq \varphi(u) + (v-u)\varphi'(v) \leq \varphi(u) + 2(v-u)/\log(e/v). \quad ∎$$

The proof of Theorem 1.1 is by induction over n. The result is rather obvious if $n=1$. We perform the induction step from $n-1$ to n.

Step 1. The key point is that it follows from Theorem 2.4 that there exists $k \leq n$ for which

$$\sum_{i \neq k} \left| \int_{A_k} r_i d\mu \int_{B_k} r_i d\mu \right| \leq K_0 \mu(A_k) \mu(B_k) \log \frac{e}{\sum_{\ell \leq n} \mu(A_\ell) \mu(B_\ell)}. \tag{4.2}$$

We assume without loss of generality that $k=n$. We define

$$A^0 = \left\{ x \in \{0,1\}^{n-1}; (x,0) \in A \right\}$$

$$A^1 = \left\{ x \in \{0,1\}^{n-1}; (x,1) \in A \right\}$$

and B^0, B^1 similarly. Since A is increasing, we have $A^0 \subset A^1$. We denote by μ' the normalized counting measure on $\{0,1\}^{n-1}$. Let $a^j = \mu'\left(A^j\right)$, $b^j = \mu'\left(B^j\right)$ for $j \in \{0,1\}$. Thus $a^0 \leq a^1, b^0 \leq b^1$. For $i < n, j \in \{0,1\}$, we set $a_i^j = \mu'\left(\left(A^j\right)_i\right)$, and we define b_i^j similarly. We wish to prove (1.2) for a certain constant K_1, that will be determined later (actually $K_1 = 2K_0$). Thus by induction hypothesis,

$$\mu'\left(A^0 \cap B^0\right) - a^0 b^0 \geq \frac{1}{K_1}\varphi\left(\sum_{i<n} a_i^0 b_i^0\right),$$

$$\mu'\left(A^1 \cap B^1\right) - a^1 b^1 \geq \frac{1}{K_1}\varphi\left(\sum_{i<n} a_i^1 b_i^1\right).$$

Since

$$\mu(A \cap B) = \frac{1}{2}\left(\mu'\left(A^0 \cap B^0\right) + \mu'\left(A^1 \cap B^1\right)\right)$$

and since φ is convex, we have

$$\mu(A \cap B) - \frac{1}{2}\left(a^0 b^0 + a^1 b^1\right) \geq \frac{1}{K_1}\varphi\left(\sum_{i<n} \frac{1}{2}\left(a_i^0 b_i^0 + a_i^1 b_i^1\right)\right).$$

Since $\mu(A) = \frac{1}{2}\left(a^1 + a^0\right), \mu(B) = \frac{1}{2}\left(b^0 + b^1\right)$, we have

$$\mu(A \cap B) - \mu(A)\mu(B) \geq \frac{1}{K_1}\varphi\left(\sum_{i<n} \frac{1}{2}\left(a_i^0 b_i^0 + a_i^1 b_i^1\right)\right) + \frac{1}{4}\left(a^1 - a^0\right)\left(b^1 - b^0\right)$$

Step 2. Thus, we are reduced to show that

$$\text{(4.3)} \quad \frac{1}{K_1}\varphi\left(\sum_{i\leq n} a_i b_i\right) \leq \frac{1}{K_1}\varphi\left(\sum_{i<n} \frac{1}{2}\left(a_i^0 b_i^0 + a_i^1 b_i^1\right)\right) + \frac{1}{4}\left(a^1 - a^0\right)\left(b^1 - b^0\right).$$

where $a_i = \mu(A_i), b_i = \mu(B_i)$. We will deduce this inequality from (4.1) with

$$u = \sum_{i<n} \frac{1}{2}\left(a_i^0 b_i^0 + a_i^1 b_i^1\right), v = \sum_{i\leq n} a_i b_i.$$

We observe that $a_i = \frac{1}{2}\left(a_i^0 + a_i^1\right)$ (and similarly for b_i). Thus

$$v - u = a_n b_n + \sum_{i<n}\left(\frac{1}{4}\left(a_i^0 + a_i^1\right)\left(b_i^0 + b_i^1\right) - \frac{1}{2}\left(a_i^0 b_i^0 + a_i^1 b_i^1\right)\right)$$

$$= a_n b_n + \frac{1}{4}\sum_{i<n}\left(a_i^1 - a_i^0\right)\left(b_i^0 - b_i^1\right).$$

We now observe that

$$a_i^1 - a_i^0 = \int_{A^1} r_i d\mu' - \int_{A^0} r_i d\mu' = \int_{A^1 \setminus A^0} r_i d\mu' = 2 \int_{A_n} r_i d\mu.$$

A similar equality for $b_i^1 - b_i^0$ yields

$$|v-u| \leq a_n b_n + \sum_{i<n} \left| \int_{A_n} r_i d\mu \right| \left| \int_{A_n} r_i d\mu \right|$$

and, combining with (4.2) we have

$$\begin{aligned} |v-u| &\leq a_n b_n + K_0 a_n b_n \log \frac{e}{\sum_{\ell \leq n} a_\ell b_\ell} \\ &\leq 2K_0 a_n b_n \log \frac{e}{\sum_{\ell \leq n} a_\ell b_\ell} = 2K_0 a_n b_n \log \frac{e}{v} \end{aligned}$$

since $\sum_{\ell \leq n} a_\ell b_\ell \leq 1$ by (2.3), and since we may assume $K_0 \geq 1$.

It then follows from (4.1) that

$$\varphi(v) \leq \varphi(u) + 2K_0 a_n b_n. \tag{4.4}$$

We now observe that

$$a^1 - a^0 = \mu'\left(A^1 \setminus A^0\right) = 2\mu(A_n) = 2a_n$$

so that (4.3) reads

$$\frac{1}{K_1}\varphi(v) \leq \frac{1}{K_1}\varphi(u) + a_n b_n$$

and indeed follows from (4.4) if $K_1 = 2K_0$. ∎

5. Sets with large boundary

Let us look at the proof of (2.2). One way to look at this proof is, setting $\beta_i = \int_A r_i d\mu, \alpha_i = \beta_i \left(\sum_{i \leq n} \beta_i^2\right)^{-1}, f = \sum_{i \leq n} \alpha_i r_i$, to write

$$\left(\sum_{i \leq n} \beta_i^2\right)^{1/2} = \int_A f d\mu \leq \mu(A)^{1/2} \left(\int f^2 d\mu\right)^{1/2} = \mu(A)^{1/2}. \tag{5.1}$$

In order to have the leftmost and the rightmost term of the same order, the two central terms must be of the same order. The natural way to make $\int_A f d\mu$ large is by requiring that A resembles a set $\left\{\sum_{i\leq n} \alpha_i r_i \geq t\right\}$. One is hence led to think that the typical example when $\mu(A) = 1/2$ is the set $\left\{\sum_{i\leq n} r_i \geq 0\right\}$. In that case, we indeed have that $\sum_{i\leq n} \mu(A_i)^2$ is of order 1. On the other hand we have, in that example, a rather special feature, namely $\bigcup_{i\leq n} A_i$ is of measure of order $1/\sqrt{n}$.

For a point x of an increasing set, let us denote by $N(x)$ its number of neighbors outside A, that is

$$N_A(x) = \text{card } \{i; x \in A_i\}.$$

We observe that, by (2.2) and Cauchy–Schwarz,

$$\text{(5.2)} \quad \int N_A(x)\, d\mu(x) = \sum_{i\leq n} \mu(A_i) \leq \sqrt{n} \left(\sum_{i\leq n} \mu(A_i)^2\right)^{1/2} \leq \sqrt{n}\mu(A)^{1/2} \leq \sqrt{n}.$$

Proposition 5.1. *For each n there exists an increasing subset A of $\{0,1\}^n$ such that $\mu\left(\left\{N_A(x) \geq \frac{\sqrt{n}}{K}\right\}\right) \geq \frac{1}{K}$.*

As is shown by (5.2), this is the best order possible, and is in sharp contrast with the case $B = \left\{\sum_{i\leq n} r_i \geq 0\right\}$, where N_B is of order n on a set of measure of order $1/\sqrt{n}$.

Proof. The proof is probabilistic. We will skip most of the routine computations.

Step 1. Consider an integer p to be determined later. Consider random variables $(Y_{ij})_{i\leq p, j\leq 2^p}$ that are independent uniform over $\{1,\ldots,n\}$. We consider the random subset A of $\{0,1\}^n$ given by

$$x \in A \iff \forall j \leq 2^p, \exists i \leq p, x(Y_{ij}) = 0.$$

Step 2. Consider now $x \in \{0,1\}^n$. Set $k = \text{card } \{i \leq n; x_i = 0\}$. We estimate, in function of k, the probability that $x \in A$. By independence, given $j \leq 2^p$, we have

$$P\left(\forall i \leq p,\, x(Y_{i,j}) = 0\right) = \left(\frac{k}{n}\right)^p$$

so that

$$(5.3)\quad P(x\in A)=\left(1-\left(\frac{k}{n}\right)^{p}\right)^{2^p}\simeq\exp\left(-\left(\frac{2k}{n}\right)^{p}\right)\simeq\exp\left(-\left(1+\frac{2t}{\sqrt{n}}\right)^{p}\right)$$

where we have defined t by $k=\frac{n}{2}+t\sqrt{n}$. A crucial feature of (5.3) is that this is of order 1 as long as $p\leq\sqrt{n}$, $t\leq 1$.

Step 3. Let us say that y is a neighbor of x if $y=T_ix$ for some i, i.e. card $\{i;x_i\neq y_i\}=1$. Consider x as in Step 1, and the event $H(x,r)$ that $x\in A$ and has r neighbors outside A. We will prove that if $|t|\leq 1$, and if p is the largest for which $p\leq\sqrt{n}$ then

$$(5.4)\qquad P\left(H\left(x,\frac{p}{K}\right)\right)\geq\frac{1}{K}.$$

Since $\mu(\{x;|t|\leq 1\})\geq\frac{1}{K}$, it follows from Fubini theorem that with positive probability we have

$$\mu\left(\left\{x;H\left(x,\frac{p}{K}\right)\right\}\right)>\frac{1}{K}$$

and this proves the result.

The proof of (5.4) is an elaboration of the argument of Step 2. We have $H(x,r)\supset\{x\in A\}\cap H^1(x,r)$ where $H^1(x,r)$ is given by

$$\exists j_1,\ldots,j_r,\forall\ell\leq r,\ \text{there exists a unique } i_\ell\leq p$$
$$\text{with } x\left(Y_{i_\ell,j_\ell}\right)=1,\ \text{and the points } Y_{i_\ell j_\ell}\ \text{are all different.}$$

For $j\leq 2^p$, consider the events

$$V_j=\text{card }\left\{i\leq p;x\left(Y_{i,j}\right)\right\}\geq 1,$$
$$U_j=\text{card }\left\{i\leq p;x\left(Y_{i,j}\right)\right\}=1.$$

Thus, $P(U_j)=p\frac{n-k}{n}\left(\frac{k}{n}\right)^{p-1}$. Since $P(V_j)=1-\left(\frac{k}{n}\right)^{p}$ is near 1, the conditional probability $P(U_j\,|\,V_j)$ is at least

$$\frac{p}{K}\left(\frac{k}{n}\right)^{p-1}\geq\frac{p}{K}2^{-p}$$

provided we assume again $|t|\leq 1$, and since $p\leq\sqrt{n}$. We observe that $\{x\in A\}=\bigcap_{j\leq 2^p}V_j$. Conditionally on $\{x\in A\}$, the events U_j are independent, so with probability close to one at least p/K of these events occur. When U_j occurs, there exists a unique index $i(j)$ such that $x\left(Y_{i(j)}\right)=1$. If we condition with respect to the set J of indexes for which U_j occurs, these indexes $i(j)$ are independent uniformly distributed in the set $I=\{i;x_i=1\}$ (of cardinality about $n/2$). Thus with probability close to one, at least $p/2K$ of these indexes are distinct.

We have shown that $P\left(H\left(x,\frac{p}{K}\right)\Big|\{x\in A\}\right)$ is close to one. Combining with the result of Step 2 finishes the proof. ∎

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